ON THE PROBLEM OF CONTROL FOR A SYSTEM OF DIFFERENTIAL EQUATIONS WITH TIME LAG

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PMM Vol.30, № 6, 1966, pp. 1121-1124

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(Received May 21, 1966)

We shall investigate the problem of bringing the motions of a controlled system described by linear differential equations with time lag, into the state of equilibri m. Let the system with time lag be

$$x(t) = Ax(t) + Gx(t-\tau) + Bu(t) \tag{1}$$

where A and G are constant $n \times n$ matrices and B is a constant $n \times m$ matrix. Function $u(t) = \{u_1(t), \ldots, u_m(t)\}$ denotes an m-dimensional control. Time lag τ is constant. Let us consider the problem of stabilization [1] of the system (1). This means, that for the system in question such a control $u(t)$ should be found which, firstly, carries the system from its given initial state $x_0(t) = \varphi(t)$, $(-\tau \leqslant t \leqslant 0)$ into the state $x(T) = 0$ and which, secondly, maintains it in this state over the interval of time $T \leq t \leq T+\tau$. (We should note, that problems of control for the systems with time lag were investigated in their various aspects in [2 to 4]).

Let us consider one of the simplest cases. Assume, that the matrix G is nonsingular. Let the vectors $b^{(i)}$ $(i = 1, \ldots, m)$ denote the columns of matrix B and let us write the equation $Gc = b$. This defines uniquely vector c in terms of the known vector b. In particular, for each of the vectors $b^{(i)}$ we can find $c^{(i)} = G^{-1}b^{(i)}$. Now suppose, that $n \leq 2m$ and that n linearly independent vectors can be selected from the set of vectors $b^{(i)}$ and $c^{(i)}$ $(i = 1, \ldots, m)$. We can assume the vectors $b^{(i)}$ to be linearly independent without any loss of generality. Then, we can include the vectors $b^{(i)}$ as first m of n linearly independent vectors and arrange them in such an order, that the linearly independent vectors will be $b^{(1)}, \ldots, b^{(m)}$, $c^{(1)}, \ldots, c^{n-m}$. Let these vectors form a base on the space (*) $[x_1, \ldots, x_n]$ so, that $b_j^{(i)} = \delta_{ij}$, $c_j^{(k)} = \delta_{k+m\,j}$; $i = 1, \ldots, m$; $k = 1, \ldots, n-m$; $j = 1, \ldots, n$ (where $b_j^{(i)}$ and $c_j^{(k)}$ are components of vectors $b^{(i)}$ and $c^{(k)}$ and δ_{ij} is a Kronecker delta). In such coordinates, matrices G and B have the form

$$G = \begin{pmatrix} G_1 & E_{n-m} \\ G_2 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} E_m \\ 0 \end{pmatrix}$$

Here E_{n-m} and E_m are $(n-m)$ and m-dimensional unit matrices and G_2 denotes a nonsingular $(m \times m)$ matrix. We shall now introduce the following additional notation. We shall denote the m-dimensional subspace generated

*) We retain the old notation for the phase coordinates in order to simplify the symbolism.

by the vectors $b^{(i)}$, (i = 1, . . ., m) by $F^{(a)}$, and the $(n-m)$-dimensional subspace generated by $c^{(k)}$, by $F^{(b)}$. The direct sum of $F^{(a)}$ and $F^{(b)}$ is, obviously, the whole of the space $\{x_1, \ldots, x_n\}$. We shall also assume that $x^{(a)}$ is an m-dimensional vector whose components are $x_i^{(a)} = x_i$ ($i = 1, \ldots, m$) while $x^{(b)}$ is an $(n-m)$-dimensional vector with components $x_j^{(b)} = x_{m+j}$ (j = 1, ..., $n-m$). In the latter case n, a vector with components $(x_1^{(a)}, \ldots, x_m^{(a)}, 0, \ldots, 0)$ will denote the component of x belonging to $F^{(a)}$, and $(n-m)$, a vector with components $(0, \ldots, 0, x_1^{(b)}, \ldots, x_{n-m}^{(b)})$ will denote the component of x belonging to $F^{(b)}$. The $(n-m)$-dimensional vector with components $(u_1, \ldots, u_{n-m})$ obtained from u will be denoted by $u^{(1)}$, and the $(2m-n)$-dimensional vector with components $(u_{n-m+1}, \ldots, u_m)$, by $u^{(2)}$.

Then, $u_i^{(1)} = u_i$ ($i = 1, \ldots, n-m$) and $u_j^{(2)} = u_{j+n-m}$ (j = 1, $2m-n$).

Finally, let us put

$$A = \begin{pmatrix} A^{(1)}, & A^{(3)} \\ A^{(2)}, & A^{(4)} \end{pmatrix} \qquad G_2 = \begin{pmatrix} G_2{}^{(1)} \\ G_2{}^{(2)} \end{pmatrix}$$

Here $A^{(1)}$ and $A^{(4)}$ are the $(m\times m)$ and $(n-m)\times(n-m)$ matrices respectively, while $G_2^{(2)}$ is a rectangular $m\times(n-m)$ matrix. We assume that $T = \tau + \varepsilon$ ($0 < \varepsilon \leqslant \tau$). The necessary and sufficient condition for the vector $x(t)$ to be identically zero on the interval $T \leqslant t \leqslant T + \tau$, are

$$Bu(t) + Gx(t-\tau) = 0, \qquad (T \leqslant t \leqslant T+\tau) \tag{2}$$

$$x(T) = 0 \tag{3}$$

Let us write (2) as

$$u^{(1)}(t) + G_1 x^{(a)}(t-\tau) + x^{(b)}(t-\tau) = 0, \qquad u^{(2)}(t) + G_2^{(1)} x^{(a)}(t-\tau) = 0 \tag{4}$$

$$G_2^{(2)} x^{(a)}(t-\tau) = 0 \tag{5}$$

Here (5) defines the necessary and sufficient conditions for (2) to be fulfilled by a suitable choice of $u(t)$. At the same time, the form of (5) implies that (2) can be satisfied in any case, provided that $x^{(a)}(t) = 0$ when $T-\tau \le t \le T$. In other words, condition (2) is fulfilled, when the vector $x(t)$ is adjacent to all t from the interval $T-\tau \le t \le T$ in the subspace $F^{(b)}$ If the last requirement is satisfied, then the conditions (4) can yield the control $u(t)$ on the interval $T \le t \le T+\tau$, namely

$$u^{(1)}(t) = -x^{(b)}(t-\tau), \qquad u^{(2)}(t) = 0$$

From this it follows, that when $T-\tau \le t \le T$, then the vector $x(t)$ lies in $F^{(b)}$. But the last condition is fulfilled if and only if

$$x^{\cdot}(t) \in F^{(b)} \quad \text{for} \quad T-\tau \leqslant t \leqslant T, \qquad x(T-\tau) \in F^{(b)}$$

or, in the more detailed form

$$x^{\cdot(a)}(t) = A^{(3)} x^{(b)}(t) + \begin{pmatrix} G_1 & E_{n-m} \\ G_2{}^{(1)} & 0 \end{pmatrix} x(t-\tau) + u(t) = 0 \tag{6}$$

$$x^{\cdot(b)}(t) = A^{(4)} x^{(b)}(t) + G_2^{(2)} x^{(a)}(t-\tau) \qquad (T-\tau \leqslant t \leqslant T)$$

$$x^{(a)}(T-\tau) = x^{(a)}(\varepsilon) = 0 \tag{7}$$

Equation (6) allows us to find $u(t)$ on the interval $T-\tau \le t \le T$.

It remains now to establish the control $u(t)$ on the interval $0 \le t \le \varepsilon$ such, that conditions (3) and (7) are fulfilled. Let us write the condition (7) first. We note that for $0 \le t \le \varepsilon$, Equation (1) has the form

$$x^{\cdot}(t) = Ax(t) + G\varphi(t-\tau) + Bu(t) \qquad (x(0) = \varphi(0)) \tag{8}$$

Using the Cauchy formula [5] we find, that the equality $x^{(a)}(\varepsilon) = 0$ is equivalent to

$$\int_0^\varepsilon X^{(1)}(\varepsilon-\vartheta)\, u(\vartheta)\, d\vartheta = \gamma^{(a)} \tag{9}$$

Here $X^{(1)}(t-\vartheta)$ denotes an m-dimensional square matrix related to the fundamental matrix $X(t-\vartheta)$ $(X(0)=E)$ of the n-dimensional homogeneous linear system

$$x^{\cdot}(t)=Ax(t)$$

in such a way, that

$$X(t-\vartheta)=\begin{pmatrix} X^{(1)}(t-\vartheta) & X^{(3)}(t-\vartheta) \\ X^{(2)}(t-\vartheta) & X^{(4)}(t-\vartheta)\end{pmatrix}$$

The magnitude $\gamma^{(a)}$ is an m-dimensional constant vector given by

$$\gamma^{(a)}=-\left[(X^{(1)}(\varepsilon)X^{(3)}(\varepsilon)),\varphi(0)+\int_0^{\varepsilon}(X^{(1)}(\varepsilon-\vartheta),X^{(3)}(\varepsilon-\vartheta))\,G\varphi(\vartheta-\tau)\,d\vartheta)\right]$$

Let us now consider condition (3). In view of (6) and (7), it means that $x^{(b)}(T)=x^{(b)}(\tau+\varepsilon)=0$. We can determine $x^{(b)}(\tau+\varepsilon)$ using the Cauchy formula, but we must remember that the vector $x(t)$ satisfies the condition (8) on the interval $0\le t\le\varepsilon$ and the vector $x^{(b)}(t)$ satisfies the second equation of (6) on the interval $\varepsilon\le t\le\tau+\varepsilon$. But then, writing out, on one hand, the complete expression for $x^{(b)}(\tau+\varepsilon)$ and assuming, on the other hand, that $x^{(b)}(\tau+\varepsilon)=0$, we find, that (3) reduces to

$$X^{(b)}(\tau)\int_0^{\varepsilon}X^{(2)}(\varepsilon-\vartheta)\,u(\vartheta)\,d\vartheta+\int_0^{\varepsilon}X^{(b)}(\varepsilon-\xi)\,G_2{}^{(2)}\left[\int_0^{\xi}X^{(1)}(\xi-\vartheta)\,u(\vartheta)\,d\vartheta\right]d\xi=\gamma^{(b)} \tag{10}$$

Here $X^{(b)}(t-\vartheta)$ $(X^{(b)}(0)=E)$ denotes the fundamental matrix of the $(n-m)$-dimensional system

$$x^{\cdot(b)}(t)=A^{(4)}x^{(b)}(t)$$

while the constant $(n-m)$ vector $\gamma^{(b)}$ can be found from

$$-\gamma^{(b)}=X^{(b)}(\tau)(X^{(2)}(\varepsilon),X^{(4)}(\varepsilon)\,\varphi^{(b)}(0)+$$

$$+X^{(b)}(\tau)\int_0^{\varepsilon}(X^{(2)}(\varepsilon-\vartheta),X^{(4)}(\varepsilon-\vartheta))\,G\varphi(\vartheta-\tau)\,d\vartheta+$$

$$+\int_0^{\tau}X^{(b)}(\tau+\varepsilon-\vartheta)\,G_2{}^{(2)}\varphi^{(a)}(\vartheta-\tau)\,d\tau+\int_0^{\varepsilon}X^{(b)}(\varepsilon-\vartheta)\,G_2{}^{(2)}(X^{(1)}(\vartheta),X^{(3)}(\vartheta))\,\varphi(0)\,d\vartheta+$$

$$+\int_0^{\varepsilon}X^{(b)}(\varepsilon-\vartheta)\,G_2{}^{(2)}\int_0^{\vartheta}(X^{(1)}(\vartheta-\xi),X^{(3)}(\vartheta-\xi))\,G\varphi(\xi-\tau)\,d\xi\Big]\,d\vartheta$$

Equations (9) and (10) can be combined into

$$\int_0^{\varepsilon}h^{(i)}(\varepsilon,\vartheta)\,u(\vartheta)\,d\vartheta=\gamma_i$$

Here $h^{(i)}(\varepsilon,\vartheta)$, $(i=1,\ldots,n)$ denote k-dimensional column vectors coinciding with the corresponding columns of the matrix $X^{(1)}(\varepsilon-\vartheta)$ for $i=1,\ldots,m$, and the matrix

$$X^{(b)}(\tau)\,X^{(2)}(\varepsilon-\vartheta)+\int_{\vartheta}^{\varepsilon}X^{(b)}(\varepsilon-\xi)\,G_2{}^{(2)}X^{(1)}(\xi-\vartheta)\,d\xi$$

for $i=m+1,\ldots,n$. Magnitudes γ_i are pure numbers, and

$$\gamma_i^{(a)}=\gamma_i;\quad \text{if } i=1,\ldots,m,\quad \gamma_j^{(b)}=\gamma_{j+m},\quad \text{if } j=1,\ldots,n-m$$

In this manner we have reduced our problem to the problem of momentums [6]. It has for any γ_i a solution, if and only if the vectors $h^{(i)}(\varepsilon,\vartheta)$ are linearly independent. Control can then be obtained by well known methods, and the problem has a unique solution. Any function $u^{\circ}(t)$ can be singled out of the set of solutions by imposing on $u(t)$ additional constraints such as some conditions of optimality, for example in form of a minimum of some norm of $u(t)$. The latter can be achieved by standard methods [6]. Finally, wewe shall note that in our case the functions $h^{(i)}(\varepsilon, \vartheta)$ will always be linearly independent, provided that $|A| \neq 0$ and that the conditions of generality of position [7] hold for the matrices A and B .

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Translated by L.K.